Assignment 1 Q10.

Two lines are skew

iff $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \neq 0$

Two direction

($\Rightarrow$)

first logic: if det $\neq 0$, then two lines are skew

second logic: suppose not skew, then det $= 0$

($\Leftarrow$)

You also need to show this in your work.

Only talk about this in tutorial class

($\Rightarrow$): Suppose

$$L_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

$$L_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

are skew.

$$\mathcal{L}_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \quad t \in \mathbb{R}$$

$$\mathcal{L}_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + s\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \quad s \in \mathbb{R}$$

L_1 and L_2 are skew :

i.e. $L_1 \not\parallel L_2$ and no intersection

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \not\parallel \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \text{ and } \boxed{\text{no } \cap}$$

$\nexists\ s,t \in \mathbb{R}$ so that

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + s \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

$\Leftrightarrow \nexists\ s,t \in \mathbb{R}$ so that

$$\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} = t \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} - s \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} a_1 & a_2 & x_2 - x_1 \\ b_1 & b_2 & y_2 - y_1 \\ c_1 & c_2 & z_2 - z_1 \end{array}\right) \text{ is inconsistent } \checkmark$$

$$\det \begin{pmatrix} a_1 & -a_2 & x_2 - x_1 \\ b_1 & -b_2 & y_2 - y_1 \\ c_1 & -c_2 & z_2 - z_1 \end{pmatrix} = 0 \quad \text{wrong} \quad \times$$

Back to linear algebra concept:

$\nexists\ s,t \in \mathbb{R}$ so that

$$\left.\begin{pmatrix} x_2-x_1 \\ y_2-y_1 \\ z_2-z_1 \end{pmatrix} = t\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} - s\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}\right\} \Leftarrow$$

Since $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \not\parallel \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$, $\leftarrow \because$ two lines not parallel

$\left\{\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}\right\}$ is linearly independent

Now $\begin{pmatrix} x_2-x_1 \\ y_2-y_1 \\ z_2-z_1 \end{pmatrix}$ is $\boxed{\text{NOT}}$ in the span of

$\left\{\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}\right\}$.

So $\left\{\begin{pmatrix} x_2-x_1 \\ y_2-y_1 \\ z_2-z_1 \end{pmatrix}, \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}\right\}$ is linearly independent

From this you can conclude that

$$\det\begin{pmatrix} x_2-x_1 & a_1 & a_2 \\ y_2-y_1 & b_1 & b_2 \\ z_2-z_1 & c_1 & c_2 \end{pmatrix} \neq 0.$$

Thm: Given an $n \times n$ matrix A,

$$A\vec{x} = 0$$

has a unique sol^{n} iff

$\det A \neq 0$

otherwise, it has infinitely many sol^{n}

Thm: For the same A, and a vector

$\vec{b}$ ($n \times 1$ matrix),

then $A\vec{x} = \vec{b}$ (consistent)

has a unique sol^{n} iff $\det A \neq 0$

Suppose $\det A = 0$, then

you have one of the two cases

Case (i) No solution (inconsistent)

(ii) Infinitely many solution (consistent)

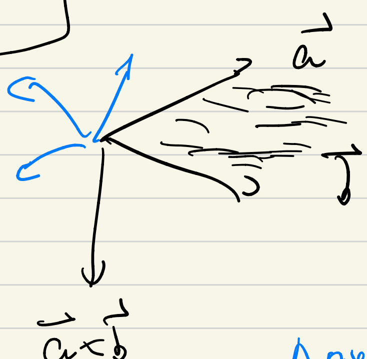
Idea of Q9:

Please don't write these things!

$$(\vec{a}\times\vec{b})\times\overbrace{(\vec{c}\times\vec{d})}^{\vec{v}} = (\vec{a}\times\vec{b})\times\vec{v}$$

suppose $\vec{a}$, $\vec{b}$ are linearly independent in $\mathbb{R}^3$

Then $\{\vec{a}, \vec{b}, \vec{a}\times\vec{b}\}$ is a basis



$$(\vec{a}\times\vec{b})\times\vec{v}$$
$$\parallel$$
$$m\vec{a}+n\vec{b}+l(\vec{a}\times\vec{b})$$

why $l=0$??

Any vectors perpendicular to $\vec{a}\times\vec{b}$ should be on the plane generated by $\vec{a}$ and $\vec{b}$

Ex 1.

Suppose that $\Pi_1 : x + y + z = 1$ and $\Pi_2 : x - y + z = 2$ are two planes in $\mathbb{R}^3$.

(a) Show that the intersection of Π_1 and Π_2 is a straight line and find a parametric equation of that line.

(b) Find the equation(s) of the plane(s) containing all the points which are equidistant from Π_1 and Π_2.

1(a) Normal vector of $\Pi_1 = (1, 1, 1)$

Normal vector of $\Pi_2 = (1, -1, 1)$

Since they are not parallel, the intersection of the two planes is a straight line.

To find the equation of the line, we need to solve the system of linear equations.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 1 \end{array}\right)$$

$R_2' = R_2 \times (-\frac{1}{2})$, $R_1' = R_1 - R_2'$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \end{array}\right) \text{ — } ①$$

pivot variable

We choose z to be the free variable

By ①,

$$x = \frac{3}{2} - z$$

$$y = -\frac{1}{2}$$

$$z = z$$

Therefore, the parametric equation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{3}{2} - z \\ -\frac{1}{2} \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}}_{\text{a point that the line passes through}} + z \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\text{direction of the line}}$$

1(b) Fix a point (x_0, y_0, z_0) in space $\mathbb{R}^3$, its distance to Π_1 is

$$\left|\frac{x_0+y_0+z_0-1}{\sqrt{1^2+1^2+1^2}}\right| = \frac{1}{\sqrt{3}}\left|x_0+y_0+z_0-1\right|$$

Similarly, its distance to Π_2 is

$$\left|\frac{x_0-y_0+z_0-2}{\sqrt{1^2+1^2+1^2}}\right| = \frac{1}{\sqrt{3}}\left|x_0-y_0+z_0-2\right|$$

That is,

$$x_0+y_0+z_0-1 = x_0-y_0+z_0-2 \quad \text{—} \ 2$$

or

$$x_0+y_0+z_0-1 = -(x_0-y_0+z_0-2) \quad \text{—} \ 3$$

From 2, $y_0 = -\frac{1}{2}$

From 3, $2x_0+2z_0 = 3$

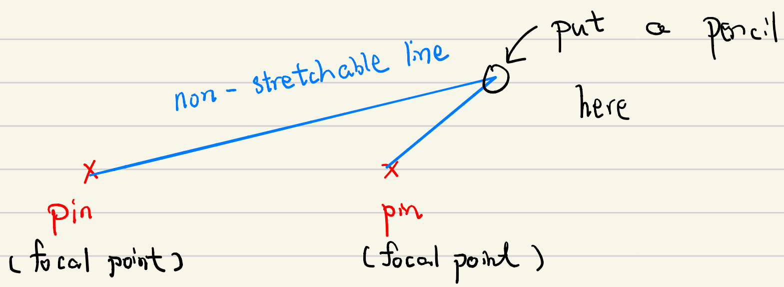
$\Pi_3 : y = -\frac{1}{2}$, $\Pi_4 : 2x+2z = 3$

are all possible planes so that every point on these planes are equidistant from Π_1 and Π_2.

Some parametric curve

Example 1 : Ellipse.

(Wiki) : An ellipse is a plane curve surrounding two focal points, so that for all points on the curve, the sum of the two distances to the focal points is a constant.



As you move the pencil, keeping the line tight, you would draw an ellipse, and the sum of distances is the length of the non-stretchable line.

An ellipse with focal points on the x-axis is given by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

or

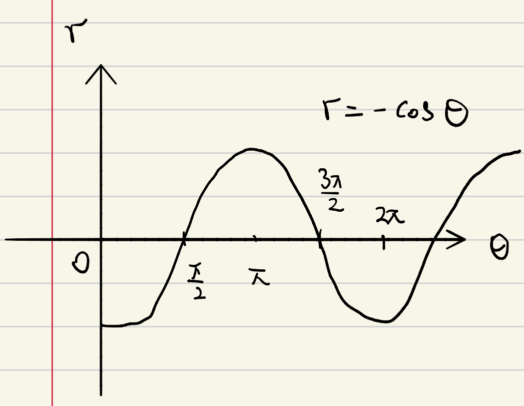
$$(x, y) = (a\cos t, b\sin t), \quad t \in [0, 2\pi]$$

parametric form

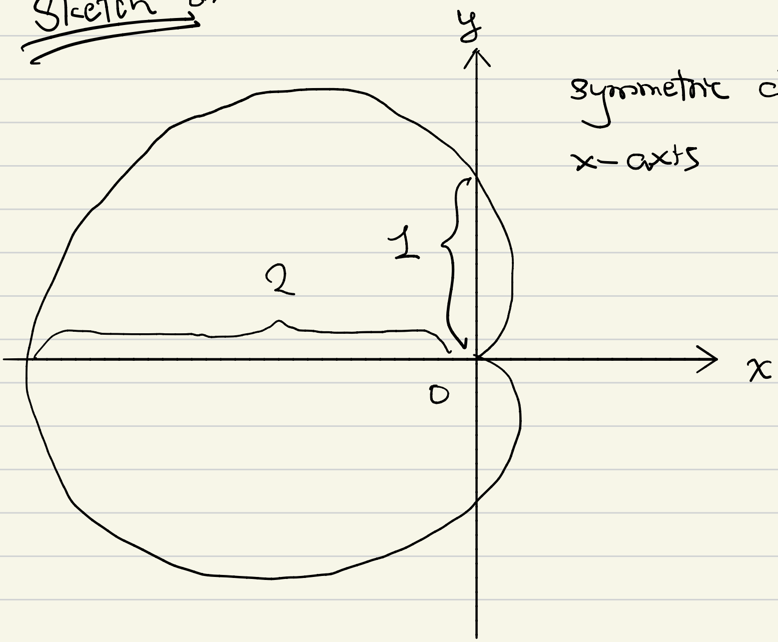
Example 2 (Cardioid)

Sketch the parametric curve $r = 1 - \cos\theta$

polar coordinate



Sketch with hand shake :



symmetric about the x-axis

For a more detailed sketching,

note $y = r\sin\theta = (1-\cos\theta)\sin\theta$

$$= \sin\theta - \frac{1}{2}\sin 2\theta$$

$$\frac{dy}{d\theta} = \cos\theta - \cos 2\theta$$

For $\frac{dy}{d\theta} = 0$,

$$\cos\theta - \cos 2\theta = 0$$

$$\cos\theta - (2\cos^2\theta - 1) = 0$$

$$2\cos^2\theta - \cos\theta - 1 = 0$$

$$\cos\theta = \frac{1 \pm \sqrt{1-4(2)(-1)}}{2(2)}$$

$$= 1 \quad \text{or} \quad -\frac{1}{2}$$

$$\theta = 0 \quad \text{or} \quad \frac{2\pi}{3}$$

$$(\text{for} \quad 0 \le \theta \le \pi)$$

When $\theta = \frac{2\pi}{3}$, $(x, y) = ??$ (Ex.)

Similarly, $x(\theta) = \cos\theta - \cos^2\theta$

$$\frac{dx}{d\theta} = \sin 2\theta - \sin\theta$$

$$\frac{dx}{d\theta} = 0 \quad \text{iff} \quad \theta = \frac{\pi}{3}, 0 \text{ or } \pi.$$

$$(\text{for} \quad 0 \le \theta \le \pi)$$

When $\theta = \frac{\pi}{3}$, $(x, y) = ??$ (Ex.)

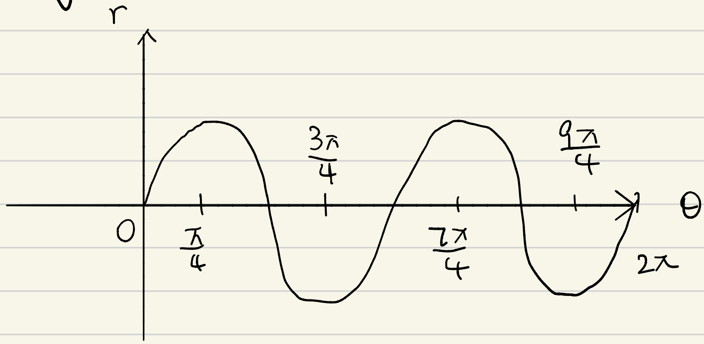
$$r = 1 - \cos\left(\theta - \frac{\pi}{2}\right) = 1 - \sin\theta$$

would be the counterclockwise rotation of the original graph about angle $\frac{\pi}{2}$.

Example 3. (Roses)

$$r = a\sin(k\theta) \quad \text{or} \quad r = a\cos(k\theta)$$

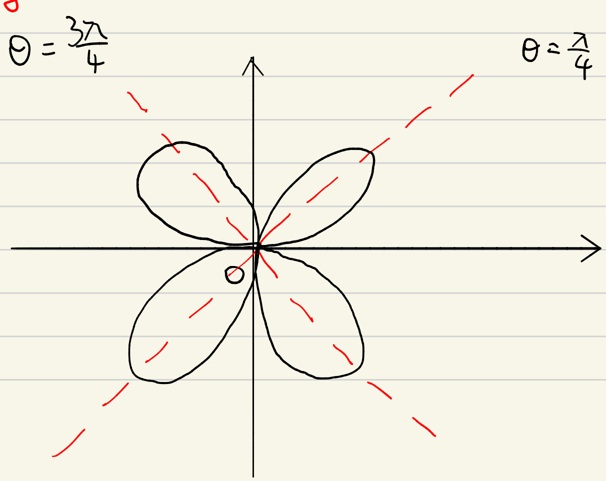
Eg. $r = 2\sin(2\theta)$



We adopt the convention that

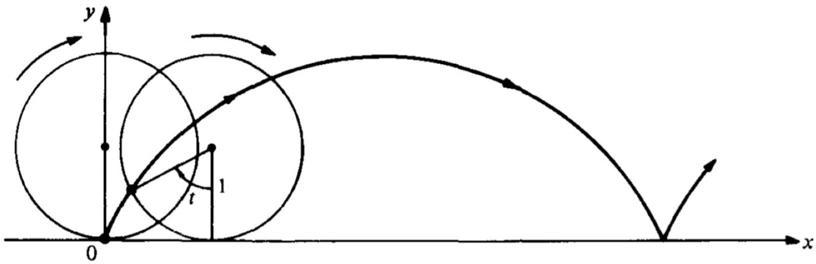
$$x = r\cos\theta, \quad y = r\sin\theta$$

(r can be negative, but (x,y) should satisfy the formula above.)



Ex 2.

In the following diagram, a circular disk of radius 1 in the plane xy rolls without slipping along the x-axis and the curve is the locus of a fixed point on the circumference which is called a *cycloid*.



(a) Give a parametrization of the cycloid.

(b) Find the arc length of the cycloid corresponding to a complete rotation of the disk.

2(a) We may decompose the movement of the fixed point into two motions.

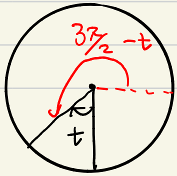
① Movement of the center of circle

② The position of that point with respect to the center of circle

① : $(x, y) = (t, 1)$

② : $(x, y) = \left(\cos\left(\frac{3\pi}{2} - t\right), \sin\left(\frac{3\pi}{2} - t\right)\right)$

$= (-\sin t, -\cos t)$



$\therefore$ Parametrization of the cycloid is $(t - \sin t, 1 - \cos t)$

(b)

Let $\vec{r}(t) = (t - \sin t, 1 - \cos t)$

$$\vec{r}\,'(t) = (1 - \cos t, \sin t)$$

$$|\vec{r}\,'(t)| = \sqrt{(1 - \cos t)^2 + (\sin t)^2}$$

$$= \sqrt{2 - 2\cos t}$$

$1 - \cos 2\theta = 2\sin^2\theta$

$$= 2\left|\sin \frac{t}{2}\right|$$

$$\therefore \int_0^{2\pi} |\vec{r}\,'(t)|\, dt = \int_0^{2\pi} 2\sin\frac{t}{2}\, dt$$

$$= \left[-4\cos\frac{t}{2}\right]_{t=0}^{2\pi}$$

$$= 8 \quad \#$$

You may check that the arclength of the Cardioid in example 2 is also 8.